RAMSEY THEORY ON GENERALIZED BAIRE SPACE

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Abstract. We show that although the Galvin-Prikry Theorem does not hold on generalized Baire space with the standard topology, there are similar theorems which do hold on generalized Baire space with certain coarser topologies.

1. Introduction and Some Definitions

Given ordinals κ and γ, $[\kappa]^\gamma$ is the set of all subsets of κ of order type γ, and $[\kappa]^{<\gamma}$ is the set of all subsets of κ of order type $< \gamma$. In this paper, for an infinite cardinal κ, we will consider colorings of $[\kappa]^\kappa$ as opposed to $[\kappa]^\mu$ for some $\mu < \kappa$. Given a function $c : X \to Y$ and a set $Z \subseteq X$, $c``Z$ is the image of Z under c. We use the convention that natural numbers are ordinals, so for example $2 = \{0, 1\}$. We will sometimes use the notation (α, β) for the set of all ordinals γ such that $\alpha < \gamma < \beta$, and $(\alpha, \beta]$ for the set $(\alpha, \beta) \cup \{\beta\}$, etc.

Definition 1.1. Let κ be a cardinal. Given sets $A, B \subseteq \kappa$, a pair (A, B) such that $A \cap B = \emptyset$ is called a *pattern*. Given $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\kappa)$, an $(\mathcal{A}, \mathcal{B})$-*pattern* is a pair (A, B) such that $A \in \mathcal{A}$ and $B \in \mathcal{B}$. A set $X \in [\kappa]^\kappa$ *matches* the pattern (A, B) iff $A \subseteq X$ and $B \cap X = \emptyset$. Finally, $[A; B]$ is the set of all $X \in [\kappa]^\kappa$ which match (A, B).

Definition 1.2. Fix $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\kappa)$. $\Sigma(\mathcal{A}, \mathcal{B})$ is the collection of all $\mathcal{S} \subseteq [\kappa]^\kappa$ that are unions of sets of the form $[A; B]$ for $(A, B) \in \mathcal{A} \times \mathcal{B}$. That is, sets $\mathcal{S}$ for which there exists a set $\mathcal{Q}$ of $(\mathcal{A}, \mathcal{B})$-patterns such that $\mathcal{S} = \{X \in [\kappa]^\kappa : X \text{ matches some } (A, B) \in \mathcal{Q}\}$. We say that $\mathcal{Q}$ *generates* $\mathcal{S}$. $\Delta(\mathcal{A}, \mathcal{B})$ is the collection of all $\mathcal{S} \subseteq [\kappa]^\kappa$ such that $\mathcal{S}$ and $[\kappa]^\kappa - \mathcal{S}$ are in $\Sigma(\mathcal{A}, \mathcal{B})$.

Hence, $\mathcal{S} \in \Sigma(\mathcal{A}, \mathcal{B})$ iff there is a collection of patterns $\{(A_i, B_i) \in \mathcal{A} \times \mathcal{B} : i \in I\}$ such that for each $X \in [\kappa]^\kappa$, $X \in \mathcal{S}$ iff $(\exists i \in I)$ X matches (A_i, B_i). Also, $\mathcal{S} \in \Delta(\mathcal{A}, \mathcal{B})$ iff there are sets $\mathcal{Q}^+, \mathcal{Q}^-$ of $(\mathcal{A}, \mathcal{B})$-patterns such that for each $X \in [\kappa]^\kappa$, $X \in \mathcal{S}$ iff X matches some $(A, B) \in \mathcal{Q}^+$, and $X \notin \mathcal{S}$ iff X matches some $(A, B) \in \mathcal{Q}^-$.

If $\mathcal{A}$ and $\mathcal{B}$ are closed under finite unions, then $\Sigma(\mathcal{A}, \mathcal{B})$ is a topology: it is closed under finite intersections and arbitrary unions, and has both

$\emptyset$ and $[\kappa]^\kappa$ as elements. If $\Sigma(\mathcal{A},\mathcal{B})$ is a topology, then $\Delta(\mathcal{A},\mathcal{B})$ is the collection of clopen sets in this topology. $\Sigma([\kappa]^{<\kappa},[\kappa]^{<\kappa})$ is the standard topology on generalized Baire space of height κ.

Definition 1.3. A collection $\mathcal{S} \in [\kappa]^\kappa$ is *Ramsey as witnessed by* $H \in [\kappa]^\kappa$ iff one of the following holds:

1) $(\forall X \in [H]^\kappa)\, X \in \mathcal{S}$;
2) $(\forall X \in [H]^\kappa)\, X \notin \mathcal{S}$.

We also say that H is *homogeneous* for $\mathcal{S}$. More generally, we say that $c : [\kappa]^\kappa \to \lambda$ is Ramsey just in case there is a set $H \in [\kappa]^\kappa$ such that $|c``[H]^\kappa| = 1$, and we say that H is homogeneous for c.

One of the earliest results in this area is the Galvin-Prikry Theorem [2], which says that not only is every open set in the topology $\Sigma([\omega]^{<\omega},[\omega]^{<\omega})$ Ramsey, but every Borel set in this topology is Ramsey as well. Next, Silver [6] showed that every analytic set in the topology $\Sigma([\omega]^{<\omega},[\omega]^{<\omega})$ is Ramsey. Ellentuck generalized this further [1] by showing that every analytic $\mathcal{S}$ in the topology $\Sigma([\omega]^{<\omega},[\omega]^{\le\omega})$ is Ramsey. Assuming the Axiom of Choice, there exists a set $\mathcal{S} \subseteq [\omega]^\omega$ that is not Ramsey. Moreover, Silver [6] showed that it is consistent with ZFC that there is a logically simple, in fact Δ^1_2, set $\mathcal{S} \subseteq [\omega]^\omega$ that is not Ramsey. On the other hand [3], if we assume the existence of large cardinals, then every $\mathcal{S} \subseteq [\omega]^\omega$ that is in $L(\mathbb{R})$ is Ramsey, where $L(\mathbb{R})$ is the smallest model of ZF that contains $\mathbb{R}$ and all the ordinals. Let us also mention that Shelah [5] has shown that if κ is a Ramsey cardinal and $c : [\kappa]^\omega \to 2$ is Borel in a certain topology, then there is a set $H \in [\kappa]^\kappa$ such that $|c``[H]^\omega| = 1$.

It is natural to ask what sets $\mathcal{S} \subseteq [\kappa]^\kappa$ for $\kappa > \omega$ are Ramsey. The standard argument that there is a set $\mathcal{S} \subseteq [\omega]^\omega$ that is not Ramsey shows that when $\kappa > \omega$, there is a set $\mathcal{S} \subseteq [\kappa]^\kappa$ in $\Delta([\kappa]^\omega,[\kappa]^{<\kappa})$ that is not Ramsey (see Proposition 6.2). In Section 2 we make the main contribution of this paper and show that when $\gamma < \kappa$, then all $\Delta([\kappa]^{<\gamma},[\kappa]^{<\gamma})$ sets are Ramsey. It is open whether Δ can be replaced with Σ.

Then, when we increase the $\mathcal{B}$ component of the patterns to include all size $< \kappa$ sets, we must simultaneously decrease the $\mathcal{A}$ component. In Section 3, we show that the following are equivalent for a cardinal $\kappa > \omega$:

- κ is weakly compact;
- All $\Delta([\kappa]^2,[\kappa]^{<\kappa})$ sets are Ramsey;
- All $\Sigma([\kappa]^2,[\kappa]^{<\kappa})$ sets are Ramsey;
- $(\forall n \in \omega)$ all $\Sigma([\kappa]^n,[\kappa]^{<\kappa})$ sets are Ramsey;

The main technique of the section is a shrinking procedure. Here is the basic version: fix a set of $([\kappa]^2, [\kappa]^{<\kappa})$-patterns $\mathcal{Q}$ and a set $H \in [\kappa]^\kappa$ such that for each $A \in [H]^2$, there is some B_A such that $(A, B_A) \in \mathcal{Q}$. Then there is some $H' \in [H]^\kappa$ such that for all distinct $a_1, a_2 \in H'$, the first element of H' greater than a_1 and a_2 is also greater than all elements of $B_{\{a_1,a_2\}}$. Each $X \in [H']^\kappa$ will match (A, B_A), where A is the set of the first two elements of X. We will modify this procedure in the following section.

In Section 4, we strengthen the $\mathcal{A}$ component of the patterns and show that if κ is a Ramsey cardinal, then all $\Sigma([\kappa]^{<\omega}, [\kappa]^{<\kappa})$ sets are Ramsey. In Section 5, we strengthen the $\mathcal{B}$ component of the patterns and show that if κ is a measurable cardinal with a κ-complete ultrafilter $\mathcal{U}$, then all $\Sigma([\kappa]^{<\omega}, \mathcal{P}(\kappa) - \mathcal{U})$ sets are Ramsey. Finally, in Section 7 we consider sets of patterns that are within L, assuming $0^\#$ exists.

2. All $\Delta([\kappa]^{<\gamma}, [\kappa]^{<\gamma})$ sets are Ramsey if $\gamma < \kappa$

Temporarily fix cardinals $\gamma < \kappa$. We call $\Sigma([\kappa]^{<\gamma}, [\kappa]^{<\gamma})$ the $<\gamma$-*box topology*; it is indeed a topology, and basic open sets are "boxes" determined by specifying membership requirements for $<\gamma$ elements of κ. We have that

$$\Sigma([\kappa]^{<\gamma}, [\kappa]^{<\gamma}) \subseteq \Sigma([\kappa]^{<\kappa}, [\kappa]^{<\kappa}).$$

It turns out that because $\Sigma([\kappa]^{<\gamma}, [\kappa]^{<\gamma})$ is so coarse, all $\Delta([\kappa]^{<\gamma}, [\kappa]^{<\gamma})$ sets are Ramsey. This follows from the next theorem:

Theorem 2.1. *Let $\gamma < \kappa$ be infinite cardinals. Let $c : [\kappa]^\kappa \to \gamma$ be continuous, where $[\kappa]^\kappa$ is given the topology $\Sigma([\kappa]^{<\gamma}, [\kappa]^{<\gamma})$ and γ is given the discrete topology. Then there is some $H \in [\kappa]^\kappa$ that is homogeneous for c, where $|\kappa - H| \leq \gamma$. If γ is a regular cardinal, we can get an H such that $|\kappa - H| < \gamma$.*

Proof. We will find a set $B \in [\kappa]^{\leq\gamma}$ such that $c \restriction [0; B]$ is constant. If γ is regular, we will have $|B| < \gamma$. Let $\langle c_\alpha : \alpha < \gamma\rangle$ be an enumeration of γ where each ordinal is listed γ times. We will construct $A_\alpha, B_\alpha \in [\kappa]^{<\gamma}$ for $\alpha < \gamma$ such that $A_\alpha \cap B_\alpha = \emptyset$ and the sets A_α are pairwise disjoint. At stage $\alpha < \gamma$, let $B = \bigcup_{\beta<\alpha} A_\beta$. Note that $|B| \leq \gamma$, and if γ is regular, then $|B| < \gamma$. There are two possibilities.

Case 1. If $c \restriction [\emptyset; B]$ is constantly c_α, then terminate the construction.

Case 2. Fix some $X \in [\emptyset; B]$ such that $c(X) \neq c_\alpha$. Let $d_\alpha = c(X)$. Since c is continuous, fix disjoint A_α, B_α such that $X \in [A_\alpha; B_\alpha]$ and $c \restriction [A_\alpha; B_\alpha]$ is constantly d_α. Note that since $A_\alpha \subseteq X$ and $X \cap B = \emptyset$, A_α is disjoint from each A_β for $\beta < \alpha$.

We claim that the construction must terminate before stage γ. Suppose that this is not the case. Fix $X \in [\emptyset; \bigcup_{\alpha<\gamma} B_\alpha]$. Fix disjoint $A, B \in [\kappa]^{<\gamma}$ such that $X \in [A; B]$ and $c \restriction [A; B]$ is constantly $c(X)$. Only $< \gamma$ many A_α's can intersect B, because the A_α's are pairwise disjoint. Fix $\alpha < \gamma$ such that A_α is disjoint from B and $c_\alpha = c(X)$. Since $A \subseteq X$, A is disjoint from B_α. We now have that A and A_α are each disjoint from B and B_α. Thus, $(A \cup A_\alpha, B \cup B_\alpha)$ is a pattern. We now have that c is constantly $c(X)$ on $[A; B]$ and it is constantly $d_\alpha \neq c_\alpha = c(X)$ on $[A_\alpha; B_\alpha]$. But since

$$[A \cup A_\alpha; B \cup B_\alpha] \subseteq [A; B] \cap [A_\alpha; B_\alpha],$$

this is impossible. $\square$

An important fact used in the proof above is that the coloring is $\Delta([\kappa]^{<\gamma}, [\kappa]^{<\gamma})$, as opposed to just $\Sigma([\kappa]^{<\gamma}, [\kappa]^{<\gamma})$. We ask whether these more general sets are Ramsey:

Question 2.2. Let $\gamma < \kappa$ be infinite cardinals. Is every $\Sigma([\kappa]^{<\gamma}, [\kappa]^{<\gamma})$ set Ramsey? In particular, is every $\Sigma([\omega_1]^2, [\omega_1]^1)$ set Ramsey? If κ is a measurable cardinal, is every $\Sigma([\kappa]^\omega, [\kappa]^1)$ set Ramsey?

In the conclusion of the previous theorem, H satisfies $|\kappa - H| \leq \gamma$. This allows us to simultaneously homogenize $< \kappa$ sets that are all $\Delta([\kappa]^{<\gamma}, [\kappa]^{<\gamma})$.

3. All $\Sigma([\kappa]^2, [\kappa]^{<\kappa})$ sets are Ramsey iff κ is weakly compact

If κ is not a weakly compact cardinal, then there is a coloring of $[\kappa]^2$ such that there is no $H \in [\kappa]^\kappa$ all of whose pairs are the same color. The collection $\Sigma([\kappa]^2, [\kappa]^{<\kappa})$ is fine enough to allow the following:

Observation 3.1. For each pair $\{a_1, a_2\} \in [\kappa]^2$, there is a $([\kappa]^2, [\kappa]^{<\kappa})$-pattern (A, B) such that a set $X \in [\kappa]^\kappa$ matches (A, B) iff its first two elements are a_1 and a_2.

This allows us to color a set $X \in [\kappa]^\kappa$ based on its first two elements.

Proposition 3.2. *Let κ be an infinite cardinal that is not weakly compact. Then there is a set in $\Delta([\kappa]^2, [\kappa]^{<\kappa})$ that is not Ramsey.*

Proof. Since κ is not weakly compact, fix a coloring $c : [\kappa]^2 \to 2$ such that there is no $H \in [\kappa]^2$ satisfying $|c``[H]^2| = 1$. Using the observation above, let $\mathcal{S} \in \Delta([\kappa]^2, [\kappa]^{<\kappa})$ be the unique subset of $[\kappa]^\kappa$ such that for each $X \in [\kappa]^\kappa$, we have $X \in \mathcal{S}$ iff $c(\{a_1, a_2\}) = 1$, where a_1, a_2 are the first two elements of X. To see that $\mathcal{S}$ is indeed $\Delta([\kappa]^2, [\kappa]^{<\kappa})$, consider the first two elements a_1, a_2 of X. If $c(\{a_1, a_2\}) = 1$, then there is a

$([\kappa]^2, [\kappa]^{<\kappa})$-pattern which witnesses that $X \in \mathcal{S}$. If $c(\{a_1, a_2\}) = 0$, then there is a $([\kappa]^2, [\kappa]^{<\kappa})$-pattern which witnesses that $X \notin \mathcal{S}$.

One can see that given any $H \in [\kappa]^\kappa$, there are $X_1, X_2 \in [H]^\kappa$ such that $X_1 \in \mathcal{S}$ and $X_2 \notin \mathcal{S}$. Hence, $\mathcal{S}$ is not Ramsey. □

On the other hand, we will show that if κ is weakly compact, then every $\Sigma([\kappa]^2, [\kappa]^{<\kappa})$ set is Ramsey. We will use the following shrinking procedure, which we isolate here for clarity.

In the following setup, we do not actually need each $A \in [X]^n$ to have an associated B_A. All we need is that for each $X' \in [X]^\kappa$, there is some $\alpha < \kappa$ such that $A := X' \cap \alpha$ has an associated B_A. However, we will not need this generality.

Definition 3.3. Let $X \in [\kappa]^\kappa$ and $\mathcal{Q}$ be a set of patterns. Fix $n \in \omega$. Suppose for each $A \in [X]^n$ there is a set B_A such that $(A, B_A) \in \mathcal{Q}$. We say that X is *fast* for $A \mapsto B_A$ iff for each $A \in [X]^n$, the only elements of $B_A \cap X$ are $< \sup A$.

Lemma 3.4. *Let $X, \mathcal{Q}, n$ be as in the definition above, where each $A \in [X]^n$ has an associated B_A. Suppose X is fast for $A \mapsto B_A$. Then every $X' \in [X]^\kappa$ matches some pattern in $\mathcal{Q}$.*

Proof. Consider any $X' \in [X]^\kappa$. Let $A \in [X']^n$ be the first n elements of X'. Consider the set $B_A \cap X'$. The only elements of $B_A \cap X$ are $< \sup A$, so therefore the only elements of $B_A \cap X'$ are $< \sup A$. On the other hand, the only elements of X' that are $< \sup A$ are the elements of A themselves, and we have that $B_A \cap A = \emptyset$. Thus, $B_A \cap X' = \emptyset$, which shows that X' matches (A, B_A). □

To produce an $X' \in [X]^\kappa$ that is fast for $A \mapsto B_A$, we shrink X by subtracting the final parts of the B_A's from X.

Lemma 3.5. *Let $X \in [\kappa]^\kappa$, $n \in \omega$, and $\mathcal{Q}$ be a set of $([\kappa]^n, [\kappa]^{<\kappa})$-patterns. Assume that each $A \in [X]^n$ has an associated B_A such that $(A, B_A) \in \mathcal{Q}$. Then there is some $X' \in [X]^\kappa$ that is fast for $A \mapsto B_A$.*

Proof. Fix a function $f : \kappa \to \kappa$ such that for each α and $A \in [\alpha]^n$, $\sup(B_A) < f(\alpha)$. Thin down X to produce an X' that satisfies $f(A) < y$ for all $A \in [X']^n$ and $y \in X'$ such that $A < y$. This works. □

Here is the promised result.

Proposition 3.6. *Let κ be a weakly compact cardinal. Then every $\Sigma([\kappa]^2, [\kappa]^{<\kappa})$ set is Ramsey.*

Proof. Fix $\mathcal{S} \subseteq [\kappa]^\kappa$ in $\Sigma([\kappa]^2, [\kappa]^{<\kappa})$. Let $\mathcal{Q}$ be a set of $([\kappa]^2, [\kappa]^{<\kappa})$-patterns which generate $\mathcal{S}$. For each $A \in [\kappa]^2$, if there is some $B \in [\kappa]^{<\kappa}$

such that $(A, B) \in \mathcal{Q}$, then let B_A be some such B. Let $c : [\kappa]^2 \to 2$ be the following coloring:

$$c(A) := \begin{cases} 1 & \text{if } (A, B) \in \mathcal{Q} \text{ for some } B, \\ 0 & \text{otherwise.} \end{cases}$$

Since κ is weakly compact, let $H \in [\kappa]^\kappa$ be homogeneous for c. That is, all pairs from H are assigned the same color by c. If $c``[H]^2 = \{0\}$, then no subset of H can match any pattern from $\mathcal{Q}$, so we are done.

If $c``[H]^2 = \{1\}$, then each $A \in [H]^2$ has an associated B_A. Apply Lemma 3.5 to get a set $H' \in [H]^\kappa$ that is fast for $A \mapsto B_A$. By Lemma 3.4, each $X \in [H']^\kappa$ matches a pattern in $\mathcal{Q}$. □

If κ is a weakly compact cardinal, then we have in fact that for every $n \in \omega$, $\lambda < \kappa$, and $d : [\kappa]^n \to \lambda$, there is some $H \in [\kappa]^\kappa$ satisfying $|d``[H]^n| = 1$. Thus, the argument from the proposition above yields the following. It implies, in particular, that if κ is weakly compact, then every set in $\Sigma([\kappa]^n, [\kappa]^{<\kappa})$ for $n \in \omega$ is Ramsey.

Proposition 3.7. *Let κ be weakly compact and let $1 \leq \lambda < \kappa$. Let $c : [\kappa]^\kappa \to (\lambda + 1)$ be such that for each $\alpha < \lambda$, $c^{-1}(\alpha) \in \Sigma([\kappa]^n, [\kappa]^{<\kappa})$. Then c is Ramsey.*

Proof. Note that we make no requirements on the complexity of $c^{-1}(\lambda)$. For each $\alpha < \lambda$, let $\mathcal{Q}_\alpha$ be the set of $([\kappa]^n, [\kappa]^{<\kappa})$-patterns which generate $c^{-1}(\alpha)$. For each $A \in [\kappa]^n$, if there is some $B \in [\kappa]^{<\kappa}$ such that $(A, B) \in \mathcal{Q}_\alpha$ for some α, then let B_A be some such B. Note that if $(A, B_1) \in \mathcal{Q}_{\alpha_1}$ and $(A, B_2) \in \mathcal{Q}_{\alpha_2}$, then $\alpha_1 = \alpha_2$. Let $d : [\kappa]^n \to (\lambda + 1)$ be the following coloring:

$$d(A) := \begin{cases} \alpha & \text{if } (A, B) \in \mathcal{Q}_\alpha \text{ for some } B, \\ \lambda & \text{otherwise.} \end{cases}$$

Since κ is weakly compact, let $H \in [\kappa]^\kappa$ be such that $|d``[H]^n| = 1$.

If $d``[H]^n = \{\lambda\}$, then consider any $X \in [H]^\kappa$. For each $A \in [X]^n$, there is no B such that $(A, B) \in \mathcal{Q}_\alpha$ for some $\alpha < \lambda$. Hence, X is not in any $c^{-1}(\alpha)$ for $\alpha < \lambda$. Thus, $X \in c^{-1}(\lambda)$. This shows that H is homogeneous for c.

The other case is that $d``[H]^n = \{\alpha\}$ for some fixed $\alpha < \lambda$. That is, for each $A \in [H]^n$, $(A, B_A) \in \mathcal{Q}_\alpha$. Apply Lemma 3.5 to get a set $H' \in [H]^\kappa$ that is fast for $A \mapsto B_A$. By Lemma 3.4, each $X \in [H']^\kappa$ matches a pattern in $\mathcal{Q}$. □

4. All $\Sigma([\kappa]^{<\omega}, [\kappa]^{<\kappa})$ sets are Ramsey if κ is Ramsey

The results in this section are analogous to those in the previous section, so we will only sketch the proofs. Recall that κ is a Ramsey cardinal iff given any $c : [\kappa]^{<\omega} \to 2$, there is some $H \in [\kappa]^\kappa$ such that for all $n \in \omega$, $|c``[H]^n| = 1$. The following is analogous to Observation 3.1:

Observation 4.1. For $A \in [\kappa]^n$, there is a $([\kappa]^n, [\kappa]^{<\kappa})$-pattern (A, B) such that a set $X \in [\kappa]^\kappa$ matches (A, B) iff its first n elements are the elements of A.

We would like to say that if κ is not a Ramsey cardinal, then there is some $\Delta([\kappa]^{<\omega}, [\kappa]^{<\kappa})$ set that is not Ramsey. However, we know only the following assertion to be true:

Proposition 4.2. *Let κ be an infinite cardinal that is not Ramsey. Then there are $\mathcal{S}_n \in \Delta([\kappa]^n, [\kappa]^{<\kappa})$ for $n < \omega$ such that there is no $H \in [\kappa]^\kappa$ homogeneous for all $\mathcal{S}_n$.*

Proof. Let $c : [\kappa]^{<\omega} \to 2$ witness that κ is not Ramsey. Using the observation above, for each $n \in \omega$, define $\mathcal{S}_n$ so that given any $X \in [\kappa]^\kappa$, $X \in \mathcal{S}$ iff the first n elements of X are colored 1 by c. If $H \in [\kappa]^\kappa$ is a set which is homogeneous for each $\mathcal{S}_n$, then $|c``[H]^n| = 1$ for each n, which is a contradiction. $\square$

The following is a straightforward modification of Proposition 3.6:

Proposition 4.3. *Let κ be a Ramsey cardinal. Then every $\Sigma([\kappa]^{<\omega}, [\kappa]^{<\kappa})$ set is Ramsey.*

Proof. Fix $\mathcal{S} \subseteq [\kappa]^\kappa$ in $\Sigma([\kappa]^{<\omega}, [\kappa]^{<\kappa})$. Let $\mathcal{Q}$ be the set of patterns which generate $\mathcal{S}$. For each $A \in [\kappa]^{<\omega}$, if there is some $B \in [\kappa]^{<\kappa}$ such that $(A, B) \in \mathcal{Q}$, then let B_A be some such B. For each $n \in \omega$, let $c_n : [\kappa]^n \to 2$ be the following coloring:

$$c_n(A) := \begin{cases} 1 & \text{if } (A, B) \in \mathcal{Q} \text{ for some } B, \\ 0 & \text{otherwise.} \end{cases}$$

Since κ is a Ramsey cardinal, let $H \in [\kappa]^\kappa$ simultaneously homogenize each c_n.

There are two cases. The first case is that for all $n \in \omega$, $c_n``[H]^n = \{0\}$. When this happens, no $X \in [H]^\kappa$ can match any pattern $(A, B) \in \mathcal{Q}$, so H is homogeneous for $\mathcal{S}$.

The other case is that there is some fixed $n \in \omega$ such that $c_n``[H]^n = \{1\}$. Each $A \in [H]^n$ has an associated B_A. Apply Lemma 3.5 to get a set $H' \in [H]^\kappa$ that is fast for $A \mapsto B_A$. By Lemma 3.4, each $X \in [H']$ matches a pattern in $\mathcal{Q}$. $\square$

If κ is a Ramsey cardinal, then for any cardinal $\lambda < \kappa$, for any coloring $d : [\kappa]^{<\omega} \to \lambda$, there is set $H \in [\kappa]^\kappa$ such that for all $n < \omega$, $|d``[H]^n| = 1$. This gives us the following:

Proposition 4.4. *Let κ be Ramsey and let $1 \le \lambda < \kappa$. Let $c : [\kappa]^\kappa \to (\lambda + 1)$ be such that for each $\alpha < \lambda$, $c^{-1}(\alpha) \in \Sigma([\kappa]^{<\omega}, [\kappa]^{<\kappa})$. Then c is Ramsey.*

Proof. The proof is analogous to Proposition 3.7. For each $\alpha < \lambda$, let $\mathcal{Q}_\alpha$ be the set of patterns which generate $c^{-1}(\alpha)$. We let $d : [\kappa]^{<\omega} \to (\lambda+1)$ be such that $d(A) := \alpha$ if $(A, B) \in \mathcal{Q}_\alpha$ for some B, and $d(A) := \lambda$ otherwise. Note that d is well-defined. Since κ is Ramsey, let $H \in [\kappa]^\kappa$ be such that $|d``[H]^n| = 1$ for all $n \in \omega$.

There are two cases. The first case is that $d``[H]^n = \{\lambda\}$ for all $n$. In this case, it can be argued that each $X \in [H]^\kappa$ is in $d^{-1}(\lambda)$. The other case is that $d``[H]^n = \{\alpha\}$ for some fixed $n < \omega$ and $\alpha < \lambda$. In this case, H can be shrunk as before to produce $H' \in [H]^\kappa$ with the property that each $X \in [H']^\kappa$ is in $c^{-1}(\alpha)$. □

5. All $\Sigma([\kappa]^{<\omega}, \mathcal{P}(\kappa) - \mathcal{U})$ sets are Ramsey if $\mathcal{U}$ is a κ-complete ultrafilter

So far, we have said little about patterns (A, B) where $|B| = \kappa$. In this section, we will show that when κ is a measurable cardinal and when we fix a κ-complete ultrafilter on κ, sets B not in the ultrafilter are small enough to be used in patterns (A, B) that will still generate Ramsey sets. Recall that an ultrafilter $\mathcal{U}$ is *κ-complete* iff it is closed under intersections of size $< \kappa$. An ultrafilter on κ is *normal* iff it is κ-complete and moreover is closed under *diagonal* intersections.

Theorem 5.1. *Let κ be a measurable cardinal and let $\mathcal{U}$ be a normal ultrafilter on κ. Then every $\Sigma([\kappa]^{<\omega}, \mathcal{P}(\kappa) - \mathcal{U})$ set is Ramsey, as witnessed by a set $H \in \mathcal{U}$.*

Proof. Fix $\mathcal{S}$ in $\Sigma([\kappa]^{<\omega}, \mathcal{P}(\kappa) - \mathcal{U})$, and let $\mathcal{Q}$ be the set of $([\kappa]^{<\omega}, \mathcal{P}(\kappa) - \mathcal{U})$-patterns which generates it. For each $A \in [\kappa]^{<\omega}$, let $C_A \in \mathcal{P}(\kappa) - \mathcal{U}$ be some set B such that $(A, B) \in \mathcal{Q}$ if such a B exists, and let $C_A = \emptyset$ otherwise.

For each $\alpha < \kappa$, let $Y_\alpha = \bigcap\{\kappa - C_A : \max A = \alpha\} \in \mathcal{U}$. Let Y be the diagonal intersection of these Y_α's: $Y = \{\beta : \beta \in \bigcap_{\alpha<\beta} Y_\alpha\}$, which is in $\mathcal{U}$ because $\mathcal{U}$ is normal. Suppose temporarily that $A \in [Y]^{<\omega}$, $y \in Y$, and $A < y$. Let $\alpha = \max A$, so $\alpha < y$. Since $y \in Y$, by definition we have $y \in Y_\alpha$. This implies that $y \in \kappa - C_A$. Hence, $y \notin C_A$.

Now let $c : [Y]^{<\omega} \to 2$ be the coloring given by $c(A) = 1$ if $(A, C_A) \in \mathcal{Q}$, and $c(A) = 0$ otherwise. Since $Y \in \mathcal{U}$ and $\mathcal{U}$ is κ-complete, there is some $H \in [Y]^\kappa$ in $\mathcal{U}$ that is homogeneous for c. If $c``[H]^n = \{0\}$ for all $n$, then no $X \in [H]^\kappa$ matches a pattern in $\mathcal{Q}$, and we are done. If $c``[H]^n = \{1\}$ for some fixed n, then consider any $X \in [H]^\kappa$. Let A be the first n elements of X. By what we said above, any element of Y greater than $\max A$ is not in C_A. Hence, every element of X greater than $\max A$ is not in C_A. This shows that $X \cap C_A = \emptyset$. Thus, X matches the pattern $(A, C_A) \in \mathcal{Q}$. □

If $\mathcal{U}$ is not a normal ultrafilter in the above theorem but only a κ-complete ultrafilter, then we have the weaker conclusion that $H \in [\kappa]^\kappa$. This can be proved by modifying Lemma 3.5.

6. Not all $\Delta([\kappa]^\omega, [\kappa]^{<\kappa})$ sets are Ramsey if $\kappa > \omega$

It is well known that assuming the Axiom of Choice, not every subset of $[\omega]^\omega$ is Ramsey. Since $[\omega]^\omega = \Delta([\omega]^\omega, [\omega]^{\leq\omega})$, we have that not every $\Delta([\omega]^\omega, [\omega]^{\leq\omega})$ set is Ramsey. In this section, we will show that the argument for $[\omega]^\omega$ shows that when $\kappa > \omega$, not every $\Sigma([\kappa]^\omega, [\kappa]^{<\kappa})$ set is Ramsey.

Observation 6.1. Let $\kappa > \omega$ be a cardinal. For $A \in [\kappa]^\omega$, there is a $([\kappa]^\omega, [\kappa]^{<\kappa})$-pattern (A, B) such that a set $X \in [\kappa]^\kappa$ matches (A, B) iff the first ω elements of X are the elements of A.

Given sets $A, B \in [\kappa]^\kappa$, recall that $A \Delta B$ is the set $(A - B) \cup (B - A)$. This next proposition uses the Axiom of Choice.

Proposition 6.2. *Let $\kappa > \omega$ be a cardinal. There is a $\Delta([\kappa]^\omega, [\kappa]^{<\kappa})$ set that is not Ramsey.*

Proof. Given a set $X \in [\kappa]^\kappa$, let X' be the set of the first ω elements of X. Given $X_1, X_2 \in [\kappa]^\kappa$, we write $X_1 \equiv X_2$ iff 1) $\sup X_1' = \sup X_2'$ and 2) $|X_1' \Delta X_2'| < \omega$. Using the Axiom of Choice, we may pick a representative from each $\equiv$-equivalence class. Let $\mathcal{S} \subseteq [\kappa]^\kappa$ be defined such that for each $X \in [\kappa]^\kappa$, $X \in \mathcal{S}$ iff $|X' \Delta Y'|$ is even, where Y is the representative from X's $\equiv$-equivalence class. Now, given any $X_1 \in [\kappa]^\kappa$, there is some $X_2 \in [X_1]^\kappa$ such that $X_1 \in \mathcal{S}$ iff $X_2 \notin \mathcal{S}$: to produce such an X_2, simply remove the first element from X_1. □

7. Constructible Patterns

We mentioned that, assuming the Axiom of Choice, there is a subset of $[\omega]^\omega$ that is not Ramsey. However, if $\mathcal{S} \subseteq [\omega]^\omega$ is in $L(\mathbb{R})$ and we assume there are large cardinals in the universe, then $\mathcal{S}$ is Ramsey [3]. With the same large cardinal assumptions, Martin showed [3] that every $\mathcal{S} \subseteq [\omega_1]^{\omega_1}$ in $L(\mathbb{R})$ is Ramsey from the point of view of $L(\mathbb{R})$. In this section, we show results of a similar flavor: if the set of patterns $\mathcal{Q}$ used to generate a set is not too complicated, then the set $\mathcal{S}$ generated in the full universe must be Ramsey.

Recall that if $0^\#$ exists, then there is a proper class of indiscernibles $\mathcal{I} \subseteq \mathrm{Ord}$, called *Silver indiscernibles*, such that L is the Skolem hull of $\mathcal{I}$. Given a cardinal κ, let $\mathcal{I}_\kappa$ refer to $\kappa \cap \mathcal{I}$.

Lemma 7.1. *Let $A \subseteq \mathcal{I}$ be in L. Then A is finite.*

Proof. Given any countably infinite subset C of $\mathcal{I}$ and $\alpha \in \mathcal{I}$ satisfying $\sup C \leq \alpha$, $0^\#$ is the theory of L_α with constant symbols for the elements of C. If A is infinite, then within L we can define $0^\#$, which is impossible. □

We must now deal with the $\mathcal{B}$ components of our patterns.

Definition 7.2. Assume $0^\#$ exists. Let $\kappa > \omega$ be a cardinal. Let $B \subseteq \kappa$ be in L. We call B *bad* iff $\mathcal{I}_\kappa - B$ has size $< \kappa$. We call B *good* iff $\mathcal{I}_\kappa \cap B$ has size $< \kappa$.

If B is bad, then no $X \in [\mathcal{I}_\kappa]^\kappa$ can match (A, B) for any A.

Lemma 7.3. *Assume $0^\#$ exists. Let $\kappa > \omega$ be a cardinal. Let $B \subseteq \kappa$ in L be not bad. Then B is good.*

Proof. Since $0^\#$ exists, let $\alpha_0 < ... < \alpha_l < \kappa$ be indiscernibles such that whenever β_1 and β_2 are between two consecutive elements of

$$0, \alpha_0, ..., \alpha_l, \kappa,$$

then $\beta_1 \in B \cap \mathcal{I}$ iff $\beta_2 \in B \cap \mathcal{I}$. The set $(\alpha_l, \kappa) \cap \mathcal{I}_\kappa$ is either a subset of B or disjoint from B. It cannot be a subset of B because then we would have that $\mathcal{I}_\kappa - B$ has size $< \kappa$, meaning B is bad. So it must be disjoint from B, and therefore B is good. □

We now have that if $\mathcal{Q} \subseteq L$ is a set of patterns and $X \in [\mathcal{I}_\kappa]^\kappa$ matches some $(A, B) \in \mathcal{Q}$, then A is finite and B is good. Hence, the (A, B) that we must consider are essentially $([\kappa]^{<\omega}, [\kappa]^{<\kappa})$-patterns:

However, this does not imply that the set $\mathcal{S}$ generated by $\mathcal{Q}$ is Ramsey. The problem is Observation 3.1, which in a more precise form gives us that for each $A \in [\kappa]^2$, there is some $B \in [\kappa]^{<\kappa}$ such that $(A, B) \in L$ and for any $X \in [\kappa]^\kappa$, X matches (A, B) iff its first two elements are the elements of A. This gives us the following:

Observation 7.4. Let κ be an infinite cardinal that is not weakly compact. Then there is a set $\mathcal{Q} \subseteq L$ of $([\kappa]^2, [\kappa]^{<\kappa})$-patterns such that the set $\mathcal{S} \subseteq [\kappa]^\kappa$ generated by $\mathcal{Q}$ is not Ramsey.

A similar situation occurs when, more generally, κ is not a Ramsey cardinal. On the other hand, we have the following:

Proposition 7.5. *Let $\kappa > \omega$ be a Ramsey cardinal. Let $\mathcal{Q} \subseteq L$ be a set of patterns. Then the set $\mathcal{S} \subseteq [\kappa]^\kappa$ generated by $\mathcal{Q}$ is Ramsey.*

Proof. Since κ is a Ramsey cardinal, $0^\#$ exists. Consider $\mathcal{I}_\kappa$. Let $\mathcal{Q}' \subseteq \mathcal{Q}$ be the set of $(A, B) \in \mathcal{Q}$ such that A is finite and B is good. By the previous lemmas, for each $X \in [\mathcal{I}_\kappa]^\kappa$, we have $X \in \mathcal{S}$ iff X is in the set generated by $\mathcal{Q}'$. Thus, it suffices to find a set $H \in [\mathcal{I}_\kappa]^\kappa$ that is homogeneous for the set generated by $\mathcal{Q}'$. For each $n \in \omega$, let $c_n : [\kappa]^n \to 2$ be the coloring defined by $c_n(A) := 1$ if $(A, B) \in \mathcal{Q}'$ for some B, and $c_n(A) := 0$ otherwise. Since κ is Ramsey, let $H \in [\mathcal{I}_\kappa]^\kappa$ homogenize each c_n. If $c_n``[H]^n = \{0\}$ for each $n$, then no $X \in [H]^\kappa$ matches a pattern in $\mathcal{Q}'$. On the other hand, suppose $c_n``[H]^n = \{1\}$ for some fixed n. Then we may apply the usual shrinking procedure, since each B under consideration is good, to produce $H' \in [H]^\kappa$ such that every $X \in [H']^\kappa$ matches a pattern in $\mathcal{Q}'$. $\square$

Here is another way to ensure that the set generated by $\mathcal{Q} \subseteq L$ is Ramsey:

Proposition 7.6. *Assume $0^\#$ exists. Let $\kappa > \omega$ be a cardinal. Let $\mathcal{Q} \in L$ be a set of patterns. Then the set $\mathcal{S} \subseteq [\kappa]^\kappa$ generated by $\mathcal{Q}$ is Ramsey.*

Proof. Suppose $\mathcal{Q} = \rho(\vec{\alpha}_0, \vec{\alpha}_1)$, where ρ is a Skolem term and $\vec{\alpha}_0, \vec{\alpha}_1$ are finite increasing sequences of elements of $\mathcal{I}$ such that $\max(\vec{\alpha}_0) < \kappa \leq \min(\vec{\alpha}_1)$. Let $I = \mathcal{I}_\kappa \cap (\max(\vec{\alpha}_0), \kappa)$. Let $J \in [I]^\kappa$ be such that between any two elements of J there are infinitely many elements of I, and there are infinitely many elements of I before the first element of J. We will show that either $[I]^\kappa \cap \mathcal{S} = \emptyset$ or $[J]^\kappa \subseteq \mathcal{S}$.

Suppose there is some fixed $X \in [I]^\kappa \cap \mathcal{S}$. Let $(A, B) \in \mathcal{Q}$ be such that $X \in [A; B]$. Because $A \subseteq X \subseteq I$, by Lemma 7.1 A is finite. Since $B \in L$, let $B = \tau(\vec{\beta}_0, \vec{\beta}_1, \vec{\beta}_2)$ where τ is a Skolem term and $\vec{\beta}_0, \vec{\beta}_1, \vec{\beta}_2$ are finite increasing sequences of elements of $\mathcal{I}$ such that

$$\max(\vec{\beta}_0) \leq \max(\vec{\alpha}_0) < \min(\vec{\beta}_1) \leq \max(\vec{\beta}_1) < \kappa \leq \min(\vec{\beta}_2).$$

Assume that all elements of A occur in $\vec{\beta}_1$. Enumerate $\vec{\beta}_1$ in increasing order as $\vec{\beta}_1 = \langle \beta_1^i : i < n \rangle$. Let $F \subseteq n$ be such that $A = \{\beta_1^i : i \in F\}$.

Now fix $Y \in [J]^\kappa$. We must show that $Y \in \mathcal{S}$. That is, we must find $(A', B') \in \mathcal{Q}$ such that $Y \in [A'; B']$. Let A' be the first $|F|$ elements of Y. Enumerate A' as $A' = \{\gamma^i \in J : i \in F\}$. We now must enlarge A' to get a set of size n. Let $\gamma^i \in I$ for $i \in n - F$ be such that the sequence

$\vec{\gamma} = \langle \gamma^i \in I : i < n \rangle$ is strictly increasing and $\gamma^{n-1} < \min(Y - A')$. This is possible because J is sparse enough. Now let $B' = \tau(\vec{\beta}_0, \vec{\gamma}, \vec{\beta}_2)$. It remains to show that $(A', B') \in \mathcal{Q}$ and $Y \in [A', B']$.

Since $(A, B) \in \mathcal{Q}$, we have

$$(\{\beta_1^i : i \in F\}, \tau(\vec{\beta}_0, \vec{\beta}_1, \vec{\beta}_2)) \in \rho(\vec{\alpha}_0, \vec{\alpha}_1).$$

By indiscernibility, we have

$$(\{\gamma^i : i \in F\}, \tau(\vec{\beta}_0, \vec{\gamma}, \vec{\beta}_2)) \in \rho(\vec{\alpha}_0, \vec{\alpha}_1).$$

That is, $(A', B') \in \mathcal{Q}$.

Because $X \subseteq I$, there is some element of $I \cap (\beta_1^{n-1}, \kappa)$ not in B. So by indiscernibility, no element of $I \cap (\beta_1^{n-1}, \kappa)$ is in B. Again by indiscernibility, no element of $I \cap (\gamma^{n-1}, \kappa)$ is in B'. However, $Y - A' \subseteq I \cap (\gamma^{n-1}, \kappa)$, because γ^{n-1} is $< \min(Y - A')$. Because also $A' \cap B' = \emptyset$, we have that $Y \cap B' = \emptyset$. This establishes that $Y \in [A'; B']$. $\square$

This next question is natural along our line of inquiry:

Question 7.7. Does it follow from large cardinals, or is it even consistent with the Axiom of Choice, that for every set $\mathcal{Q} \in L(\mathbb{R})$ of $([\omega_1]^{<\omega_1}, [\omega_1]^{<\omega_1})$-patterns, the set generated by $\mathcal{Q}$ is Ramsey?

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