Disjoint ∞-Borel Functions

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Motivating Statement

Consider the following statement:

Each $A \subseteq \mathbb{R}$ of size 2^ω can be surjected onto $\mathbb{R}$ by a Borel function.

- It follows from $\mathrm{ZF} + \mathrm{AD}$ (as we will see soon).
- It, together with AC, implies $\mathrm{add}(\mathcal{M}) < 2^\omega$. That is, there is a size $< 2^\omega$ collection of meager sets of reals whose union is not meager.
- It holds in the *iterated perfect set model* (start with CH, then add ω_2 Sacks reals by iterated forcing). Thus, it is consistent with ZFC and $\mathrm{add}(\mathcal{M}) = \omega_1 < \omega_2 = 2^\omega$. It is open whether it is consistent with $\mathrm{ZFC} + 2^\omega > \omega_2$.

Stronger Statement

Consider the following stronger statement:

Each uncountable $A \subseteq \mathbb{R}$ can be surjected onto $\mathbb{R}$ by a Borel function.

- It implies the statement on the previous slide.
- It is false if we assume ZFC (it is certainly false if we assume $\neg\mathrm{CH}$, and if CH holds, then $\mathrm{add}(\mathcal{M}) < 2^\omega$ cannot hold).
- It follows from ZF + the statement that every uncountable set of reals has a non-empty perfect subset (which follows from $\mathrm{ZF} + \mathrm{AD}$). Proof: If $A \subseteq \mathbb{R}$ is uncountable and $P \subseteq A$ is a perfect subset, then there is a real which codes the set P, and this real can be used to define a continuous function which maps $P \subseteq A$ onto $\mathbb{R}$.
- It also (almost) follows from the statement on the next slide...

Even Stronger Statement: Ψ

This will be our focus:

Definition of Ψ

Ψ is the following statement: for each $a \in \mathbb{R}$ there is a function $f_a : \mathbb{R} \to \mathbb{R}$ such that the following hold:

- The function $(a, x) \mapsto f_a(x)$ is Borel.
- $(\forall g : \mathbb{R} \to \mathbb{R})\ \{a \in \mathbb{R} : f_a \cap g = \emptyset\}$ is countable.

Relation to Previous Statement

Recall that $\mathrm{Uniformization}$ is the fragment of AC which says that given any $R \subseteq \mathbb{R} \times \mathbb{R}$ satisfying $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\,(x, y) \in R$, then there is a function $g : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{Graph}(g) \subseteq R$.

Proposition

$\mathrm{ZF} + \mathrm{Uniformization} + \Psi$ implies that if $A \subseteq \mathbb{R}$ is uncountable, then it can be surjected onto $\mathbb{R}$ by a Borel function.

Proof: Fix an uncountable set $A \subseteq \mathbb{R}$. For each $x \in \mathbb{R}$, the function $a \mapsto f_a(x)$ is Borel. We claim that for some $x \in \mathbb{R}$, the function $a \mapsto f_a(x)$ surjects A onto $\mathbb{R}$. Suppose this is not the case. For each $x \in \mathbb{R}$, the set $Y_x := \mathbb{R} - \{f_a(x) : a \in A\}$ is non-empty. Apply $\mathrm{Uniformization}$ to get $g : \mathbb{R} \to \mathbb{R}$ such that $(\forall x \in \mathbb{R})\, g(x) \in Y_x$. Then g is disjoint from f_a for each $a \in A$, which is a contradiction because g can be disjoint from only countably many of the f_a functions.

Proving Ψ from AD^+: Part 1

Without loss of generality, we will use ${}^\omega\omega$ in place of $\mathbb{R}$. AD^+ is a statement that implies both AD and that every function $g : \mathbb{R} \to \mathbb{R}$ is ∞-Borel. Note: AD implies there is no injection of ω_1 into $\mathbb{R}$.

Definition

A set $S \subseteq {}^\omega\omega$ is ∞-Borel iff there is a formula φ and a set of ordinals $C \subseteq \mathrm{Ord}$ such that $(\forall x \in {}^\omega\omega)\, x \in S \Leftrightarrow L[C][x] \models \varphi(C, x)$.

Definition

A function $g : {}^\omega\omega \to {}^\omega\omega$ is ∞-Borel iff there is a set of ordinals $C \subseteq \mathrm{Ord}$ and a formula φ such that for all $x \in {}^\omega\omega$ and $n, m \in \omega$,

$$g(x)(n) = m \Leftrightarrow L[C][x] \models \varphi(C, x, n, m).$$

Idea: ∞-Borel functions are "nice". If there is a proper class of Woodin cardinals and $A \subseteq \mathbb{R}$ is *universally Baire*, then $L(\mathbb{R}, A) \models \mathrm{AD}^+$. Thus, if there is a proper class of Woodin cardinals and $g : \mathbb{R} \to \mathbb{R}$ is universally Baire, then g is ∞-Borel.

Proving Ψ from AD^+: Part 2

We will show that AD^+ implies Ψ. For each $a \in {}^\omega\omega$, we will define $f_a : {}^\omega\omega \to {}^\omega\omega$ as follows: Fix $a \in {}^\omega\omega$. Pick some $A \subseteq \omega$ such that $A =_T a$, A is infinite, and $A \leq_T B$ whenever B is an infinite subset of A. Such a set A is easy to construct. We actually only need A to be Δ^1_1 in every infinite subset of itself.

Let $h : A \to \omega$ be a function such that $(\forall n \in \omega)\, h^{-1}(n)$ is infinite.

We will now define $f_a : {}^\omega\omega \to {}^\omega\omega$. Fix $x = \langle x_0, x_1, ...\rangle \in {}^\omega\omega$. Let $i_0 < i_1 < ...$ be the sequence of indices listing which numbers x_i are in A. That is, each $x_{i_k} \in A$, but no other x_i is in A. Define

$$f_a(x) := \langle h(x_{i_0}), h(x_{i_1}), ...\rangle$$

If there are only finitely many x_i in A, define $f_a(x)$ to be anything.

Proving Ψ from AD^+: Part 3

Main Theorem (ZF)

Assume there is no injection of ω_1 into ${}^\omega\omega$. Let $g : {}^\omega\omega \to {}^\omega\omega$ be ∞-Borel, as witnessed by the set of ordinals $C \subseteq \mathrm{Ord}$. For each $a \in {}^\omega\omega$,

$$f_a \cap g = \emptyset \Rightarrow a \in L[C].$$

Since $L[C]$ has only countably many reals in it (because $\omega_1^{L[C]}$ injects into ${}^\omega\omega \cap L[C]$), this theorem gives us that AD^+ implies Ψ.
To prove the theorem, fix $a \in {}^\omega\omega$ and assume $a \notin L[C]$. Let $A \subseteq \omega$ be the set associated with a such that $a =_T A$ and A is computable from every infinite subset of itself. We will construct an $x \in {}^\omega\omega$, by forcing over $L[C]$, such that $f_a(x) = g(x)$.

Proving Ψ from AD^+: Part 4

Let $\mathbb{H}$ be the poset of all trees $T \subseteq {}^{<\omega}\omega$ with cofinite splitting beyond the stem. We have $T_2 \leq T_1$ iff $T_2 \subseteq T_1$. Define the stronger ordering $\leq^A$ by $T_2 \leq^A T_1$ iff $T_2 \leq T_1$ and

$$(\forall t \in \mathrm{Stem}(T_2) - \mathrm{Stem}(T_1))\, t(|t| - 1) \notin A.$$

That is, $T_2 \leq^A T_1$ means that $T_2 \leq T_1$ and *the part of the stem of T_2 not in the stem of T_1 does not hit A*. Idea: If $x \in {}^{\omega}\omega$ is the generic real being added by $\mathbb{H}$ ($x = \bigcap$ the generic filter), then if $T_2 \leq^A T_1$, then T_2 does not decide any more of $f_a(x)$ than T_1 already does. The main lemma says we can hit dense subsets of $\mathbb{H}$ without deciding more of $f_a(x)$:

Main Lemma

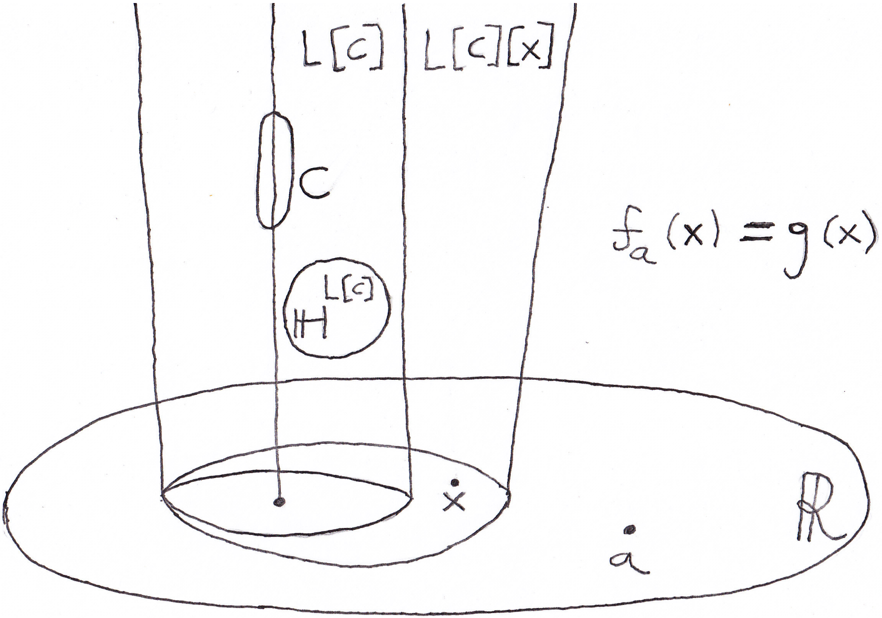
Let M be an inner model such that $A \notin M$. Let $U \in M$ be such that $U \subseteq \mathbb{H}^M$ is dense^M in $\mathbb{H}^M$. Fix $T_1 \in \mathbb{H}^M$. Then there is $T_2 \leq^A T_1$ such that $T_2 \in U$.

Proving Ψ from AD^+: Part 5

- Step 0: Let $\langle U_n : n < \omega \rangle$ be an enumeration of the dense$^{L[C]}$ subsets of $\mathbb{H}^{L[C]}$ in $L[C]$. Let $\dot{x}$ be the canonical name for the generic real $x \in {}^\omega\omega$. Let $T_0 = 1$.
- Step 1:
  - Let $T_0' \leq^A T_0$ be in U_0.
  - Let $T_0'' \leq^A T_0'$ and $m_0 \in \omega$ be such that $T_0'' \Vdash g(\dot{x})(0) = \check{m}_0$.
  - Let $T_1 \leq T_0''$ have stem 1 longer than T_0'' such that T_1 ensures that $f_a(x)(0) = m_0$.
- Step 2:
  - Let $T_1' \leq^A T_1$ be in U_1.
  - Let $T_1'' \leq^A T_1'$ and $m_1 \in \omega$ be such that $T_1'' \Vdash g(\dot{x})(1) = \check{m}_1$.
  - Let $T_2 \leq T_1''$ have stem 1 longer than T_1'' such that T_2 ensures that $f_a(x)(1) = m_1$.
- ...
- We have $(\forall i \in \omega)\, f_a(x)(i) = m_i = g(x)(i)$. Thus, $f_a(x) = g(x)$.

This completes the proof.

Proving Ψ from AD^+: Picture Summary



Corollary of the theorem

The following follows from the theorem:

Corollary

Assume there is a proper class of Woodin cardinals. Let $g : \mathbb{R} \to \mathbb{R}$ be universally Baire. Then g is disjoint from at most countably many of the functions $f_a : \mathbb{R} \to \mathbb{R}$.

Variant of the theorem

Theorem

Assume there is a proper class of Woodin cardinals. Let $\mathcal{U}$ be a selective ultrafilter on ω. Let $g : \mathbb{R} \to \mathbb{R}$ be in $L(\mathbb{R})[\mathcal{U}]$. Then g is disjoint from at most countably many of the functions $f_a : \mathbb{R} \to \mathbb{R}$.

Thus, $\mathrm{ZF}+$ there exists a non-principal ultrafilter on ω is not enough to imply $\neg\Psi$.

Open question

Earlier we showed that ZFC implies $\neg\Psi$. We ask whether the weaker statement Ψ^- is consistent with ZFC:

Definition of Ψ^-

Ψ^- is the following statement: for each $a \in \mathbb{R}$ there is a function $f^a : \mathbb{R} \to \mathbb{R}$ such that the following hold:

- The function $(a, x) \mapsto f^a(x)$ is Borel.
- $(\forall g : \mathbb{R} \to \mathbb{R})\ \{a \in \mathbb{R} : f^a \cap g = \emptyset\}$ has size $< 2^\omega$.

Note: In a model of $\mathrm{ZFC} + \Psi^-$, it must be that CH fails.

Acknowledgments

Paul Larson pointed out the argument that $\mathrm{AC} + \mathsf{add}(\mathcal{M}) = 2^\omega$ implies there is a size 2^ω set of reals that cannot be surjeccted onto $\mathbb{R}$ by a Borel function. He also explained why $L(\mathbb{R})[\mathcal{U}]$ satisfies the perfect set property, which is used in the proof that $L(\mathbb{R})[\mathcal{U}] \models \Psi$. Trevor Wilson explained the large cardinal steps in the proof that Projective Determinacy implies that every projective $g : \mathbb{R} \to \mathbb{R}$ is disjoint from at most countably many f_a's.

Thank You!

References

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